

POINT PROCESSES AND THE INFINITE SYMMETRIC GROUP. PART V: ANALYSIS OF THE MATRIX WHITTAKER KERNEL

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ABSTRACT. The matrix Whittaker kernel has been introduced by A. Borodin in Part IV of the present series of papers. This kernel describes a point process — a probability measure on a space of countable point configurations. The kernel is expressed in terms of the Whittaker confluent hypergeometric functions. It depends on two parameters and determines a J -symmetric operator K in $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$.

It turns out that the operator K can be represented in the form $L(1 + L)^{-1}$, where L is a rather simple integral operator: the kernel of L is expressed in terms of elementary functions only. This is our main result; it elucidates the nature of the matrix Whittaker kernel and makes it possible to directly verify the existence of the associated point process.

Next, we show that the matrix Whittaker kernel can be degenerated to a family of kernels expressed through the Bessel and Macdonald functions. In this way one can obtain both the well-known Bessel kernel (which arises in random matrix theory) and certain interesting new kernels.

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INTRODUCTION

The present paper continues a series of papers by Alexei Borodin and the author: [P.I – P.IV]. In [P.VI] we give a summary of the results. We refer to [P.VI] for a detailed introduction to the subject and motivation.

One of the main conclusions of our work can be stated as follows: certain stochastic point processes, which originate in harmonic analysis on the infinite symmetric group, turn out to be close to point processes arising in scaling limit of certain random matrix ensembles.

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The common feature of the point processes in question is that their correlation functions are given by determinantal expressions

$$\rho_n(x_1, \dots, x_n) = \det[K(x_i, x_j)], \quad (0.1)$$

where $K(x, y)$ is a kernel in two real variables. In principle, all the characteristics of such processes can be extracted from the corresponding kernels, though in practice this often requires a lot of work.

Random matrix theory produces a variety of interesting kernels. Among them are the Bessel kernel and the Airy kernel, which are expressed through the Bessel functions J_ν and the Airy function, respectively. About these kernels, see [F, NS, NW, TW1–3].

Our work leads to a new family of kernels, which are expressed through the Whittaker functions¹ – the *Whittaker kernel* and the *matrix Whittaker kernel*, both depending on two real parameters.

The Whittaker kernel has been introduced in [P.II] and studied in detail in [P.III]. It describes a point process on \mathbb{R}_+ whose ‘particles’ are accumulated about zero.

The matrix Whittaker kernel, which is the object of study in the present paper, has been introduced in [P.IV]. It describes a larger process, which lives on $\mathbb{R} \setminus \{0\} = \mathbb{R}_+ \cup \mathbb{R}_-$. This kernel is conveniently written as a 2×2 matrix whose entries are kernels (or integral operators) on \mathbb{R}_+ , which explains the term ‘matrix kernel’.

The matrix Whittaker kernel defines an operator in $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$ which turns out to be J -symmetric, i.e., symmetric with respect to an indefinite inner product $[f, g] = (Jf, g)$; specifically, $J = \begin{bmatrix} \text{id} & 0 \\ 0 & -\text{id} \end{bmatrix}$. Perhaps, this is the first example of a J -symmetric kernel occurring in formula (0.1) for the correlation functions.

The purpose of the present paper is to try to elucidate the nature of the matrix Whittaker kernel and its relationship to other kernels.

In section 1, we explain some general properties of the point processes governed by J -symmetric kernels on a simple model (finite state space). This enables us to avoid unnecessary technicalities in questions which are essentially based on elementary linear algebra.

In section 2, we employ the formulas of section 1 as a prompt to derive the following result about the structure of the matrix Whittaker kernel (Theorem 2.4). Let K denote the J -symmetric operator in $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$ mentioned above. We show that the operator $L = K(1 - K)^{-1}$ has the form

$$L = \begin{bmatrix} 0 & A \\ -A' & 0 \end{bmatrix}, \quad (0.2)$$

where A is a real integral operator on \mathbb{R}_+ and A' is the transposed operator. It is worth noting that the kernel of A is given by a simple expression involving no special functions at all; the Whittaker functions arise when we pass from L to K .

The passage from K to L is not a pure formal trick. As is explained in section 1, in a simpler situation of processes with finite point configurations (which is not the case of our process on $\mathbb{R}_+ \cup \mathbb{R}_-$), the operator L has a clear meaning: its kernel

¹The Whittaker functions are certain confluent hypergeometric functions. They are eigenfunctions of a second order differential operator on the semiaxis $x > 0$ and have exponential decay as $x \rightarrow +\infty$.

describes the distribution functions of the process in the whole state space. In such a situation, vanishing of the diagonal blocks of L also admits a nice interpretation (see Proposition 1.7).

For our process, the point configurations are almost surely infinite, and the basic formula expressing the distribution functions in the whole state space through the operator L (see (1.1)) becomes incorrect. However, it is tempting to combine it with the simple expression for the operator L given in Theorem 2.4 to derive meaningful conclusions about our process. In this direction, we (Borodin and I) have some conjectures but no rigorous results.

Section 3 is devoted to the spectral analysis of the matrix Whittaker kernel $K(x, y)$. We exhibit a continual basis in $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$ diagonalizing the corresponding operator K . The elements of this basis are (within a simple factor) certain Whittaker functions; they are eigenfunctions of a Sturm–Liouville differential operator. This result provides additional information about the nature of the matrix Whittaker kernel $K(x, y)$. It also provides a way to verify directly that $K(x, y)$ obeys certain conditions that ensure the existence of a point process with the correlation functions (0.1).

In section 4 we deal with the ‘tail process’. This point process is a stationary process on $\mathbb{R} \cup \mathbb{R}$. It describes (via an appropriate scaling limit) the asymptotical behavior of our initial random configurations on $\mathbb{R}_+ \cup \mathbb{R}_-$ near zero. This extends the results of sections 3–4 of [P.III].

In section 5 we study the matrix Whittaker kernel $K(x, y)$ from another point of view: hierarchy of special functions of hypergeometric type. It is well-known that the Whittaker functions can be degenerated to the Bessel functions. Employing this fact, we compute a scaling limit of the kernel $K(x, y)$. The limit transition is rather curious: we let one of the two parameters of the kernel, a , tend to infinity inside the set of the form $a_0 + 2\mathbb{Z}$ with $a_0 \in \mathbb{R}$ fixed. As a result, we get a (still two-parametric) family of matrix kernels expressed through the Bessel functions of the first kind J_ν and the modified Bessel functions of the third kind (Macdonald functions) K_ν . Taking the diagonal blocks, we get two sorts of kernels on \mathbb{R}_+ : one (expressed through the functions J_ν) is a slight generalization of the conventional Bessel kernel [F, NS, NW, TW2], and another (expressed through the Macdonald functions K_ν) is, perhaps, a new example; we called it the Macdonald kernel.

It would be interesting to understand whether the Macdonald kernel is somehow related to random matrix ensembles.

I am grateful to Alexei Borodin for numerous discussions and to Craig A. Tracy for drawing my attention to the papers [MTW], [T], [TW4].

1. A MODEL

In this section we fix a finite set \mathfrak{X} which will serve as a “state space”. We shall deal with kernels $K(x, y)$, $L(x, y)$ on $\mathfrak{X} \times \mathfrak{X}$ which will also be considered as matrices of order $|\mathfrak{X}|$ with the rows and columns indexed by the elements of \mathfrak{X} . The kernels can be real or complex. We take the counting measure on \mathfrak{X} and form the corresponding (finite-dimensional) Hilbert space $L^2(\mathfrak{X})$. Any kernel defines an operator in $L^2(\mathfrak{X})$; we shall denote it by the same letter as the kernel.

Let Ξ stand for the space of configurations in \mathfrak{X} with no multiple points. Thus, a configuration is simply a (possibly, empty) subset $\xi \subseteq \mathfrak{X}$, and the total number of configurations is equal to $2^{|\mathfrak{X}|}$.

Let $L(x, y)$ be a kernel on \mathfrak{X} . For a configuration ξ we shall denote by L_ξ the submatrix in L formed by the rows and columns from ξ , so that $\det L_\xi$ is a principal minor of L . We agree that $\det L_\emptyset = 1$.

Proposition 1.1. *Assume L is a real or complex kernel on \mathfrak{X} such that all its principal minors $\det L_\xi$ are real and nonnegative. Then there exists a probability distribution on Ξ with the weights*

$$\text{Prob}\{\xi\} = \frac{\det L_\xi}{\det(1 + L)}. \quad (1.1)$$

Note that the assumptions on L imply that the matrix $1 + L$ is invertible.

Proof. This is easy. \square

We shall consider this probability distribution as a point process on the (finite) state space \mathfrak{X} . The next step is to look at the correlation functions.

Proposition 1.2. *The correlation functions of the above process are given by the determinantal formula*

$$\rho_n(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{1 \leq i, j \leq n}, \quad (1.2)$$

where $n = 1, 2, \dots, |\mathfrak{X}|$, the points x_1, \dots, x_n are pairwise distinct, and K is given by $K = L(1 + L)^{-1}$.

Proof. An elegant proof based on the generating functional of the process is given in [DVJ], Exercise 5.4.7. \square

We shall always assume that K and L are related to each other by the transformations

$$K = L(1 + L)^{-1}, \quad L = K(1 - K)^{-1}, \quad (1.3)$$

which are well-defined provided that the matrices $1 + L$ and $1 - K$ are invertible.

Proposition 1.3. *Assume L is Hermitian nonnegative: $L = L^* \geq 0$. Then L satisfies the assumptions of Proposition 1.1 and K is Hermitian satisfying $0 \leq K < 1$. Conversely, if K is Hermitian and $0 \leq K < 1$ then the corresponding L exists and is Hermitian nonnegative.*

Proof. This follows from (1.3). \square

Thus, we dispose of a family of (finite) point processes governed by the Hermitian, nonnegative, strictly contractive kernels K . Or, equivalently, by the Hermitian nonnegative kernels L .

Henceforth we fix a partition of \mathfrak{X} into disjoint union of two subsets:

$$\mathfrak{X} = \mathfrak{X}_1 \cup \mathfrak{X}_2,$$

we decompose the Hilbert space $L^2(\mathfrak{X})$ into the direct sum

$$L^2(\mathfrak{X}) = L^2(\mathfrak{X}_1) \oplus L^2(\mathfrak{X}_2), \quad (1.4)$$

and we write any kernel (matrix) A on \mathfrak{X} in the block form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{ij} acts from $L^2(\mathfrak{X}_j)$ to $L^2(\mathfrak{X}_i)$.

We endow the space $L^2(\mathfrak{X})$ with the indefinite inner product determined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. A kernel A is called *J-Hermitian* if the corresponding operator is Hermitian with respect to the indefinite inner product. Equivalently, in terms of blocks,

$$(A_{11})^* = A_{11}, \quad (A_{22})^* = A_{22}, \quad (A_{12})^* = -A_{21}. \quad (1.5)$$

Note that if one of the kernels K, L is *J-Hermitian*, and the correspondence (1.3) makes sense, then another kernel is *J-Hermitian*, too.

Proposition 1.4. *Assume that L is *J-Hermitian* and the diagonal blocks L_{11}, L_{22} are nonnegative. Then all the principal minors $\det L_\xi$ are real nonnegative.*

Proof. Replacing L by $L + \varepsilon 1$, where $\varepsilon > 0$ is arbitrary, we may assume that the diagonal blocks L_{11}, L_{22} are strictly positive. We shall prove that under this assumption, all the principal minors $\det L_\xi$ are strictly positive. Next, replacing L by L_ξ , we remark that it suffices to prove this claim for $\det L$ only.

Thus, we have to prove that $\det L > 0$ provided that L is *J-Hermitian* and $L_{11} > 0, L_{22} > 0$. We employ the well-known formula for the determinant of a block matrix,

$$\det L = \det L_{11} \cdot \det(L_{22} - L_{21}L_{11}^{-1}L_{12})$$

which makes sense because L_{11} is invertible. We have

$$L_{11} > 0, \quad L_{22} - L_{21}L_{11}^{-1}L_{12} = L_{22} + L_{12}^*L_{11}^{-1}L_{12} > 0,$$

which implies that the both determinants are strictly positive. \square

Proposition 1.5. *In terms of blocks, the correspondence $L \leftrightarrow K$ takes the following form.*

The transform $L \mapsto K$:

$$K_{11} = (L_{11} - L_{12}(1 + L_{22})^{-1}L_{21})(1 + L_{11} - L_{12}(1 + L_{22})^{-1}L_{21})^{-1} \quad (1.6a)$$

$$K_{22} = (L_{22} - L_{21}(1 + L_{11})^{-1}L_{12})(1 + L_{22} - L_{21}(1 + L_{11})^{-1}L_{12})^{-1} \quad (1.6b)$$

$$K_{12} = (1 - K_{11})L_{12}(1 + L_{22})^{-1} \quad (1.6c)$$

$$K_{21} = (1 - K_{22})L_{21}(1 + L_{11})^{-1}. \quad (1.6d)$$

The inverse transform $K \mapsto L$:

$$L_{11} = (K_{11} - K_{12}(1 + K_{22})^{-1}K_{21})(1 + K_{11} - K_{12}(1 + K_{22})^{-1}K_{21})^{-1} \quad (1.7a)$$

$$L_{22} = (K_{22} - K_{21}(1 + K_{11})^{-1}K_{12})(1 + K_{22} - K_{21}(1 + K_{11})^{-1}K_{12})^{-1} \quad (1.7b)$$

$$L_{12} = (1 + L_{11})K_{12}(1 - K_{22})^{-1} \quad (1.7c)$$

$$L_{21} = (1 + L_{22})K_{21}(1 - K_{11})^{-1}. \quad (1.7d)$$

Here we assume that all the necessary inverse matrices exist.

Proof. Write the equality $K = L(1 + L)^{-1}$ or, equivalently, $K(1 + L) = L$ in terms of blocks:

$$K_{11}(1 + L_{11}) + K_{12}L_{21} = L_{11}$$

$$K_{11}L_{12} + K_{12}(1 + L_{22}) = L_{12}$$

$$K_{22}(1 + L_{22}) + K_{21}L_{12} = L_{22}$$

$$K_{22}L_{21} + K_{21}(1 + L_{11}) = L_{21}.$$

From this system one readily gets the relations (1.6a-d).

The second group of relations, (1.7a-d), is verified in exactly the same way. \square

Proposition 1.6. *The transforms $L \mapsto K$ and $K \mapsto L$ define a bijective correspondence between*

- (i) *the J -Hermitian kernels L such that $L_{11} \geq 0$, $L_{22} \geq 0$,*
- and*
- (ii) *the J -Hermitian kernels K such that $K_{11} < 1$, $K_{22} < 1$,*

$$K_{11} + K_{12}(1 - K_{22})^{-1}K_{21} \geq 0, \quad K_{22} + K_{21}(1 - K_{11})^{-1}K_{12} \geq 0 \quad (1.8)$$

(note that the latter two inequalities are stronger than $K_{11} \geq 0$, $K_{22} \geq 0$).

For these two sets of kernels all the inverse matrices in the formulas of Proposition 1.5 exist.

Proof. We have $1 + L_{22} > 0$, so that $1 + L_{22}$ is invertible. Further, the matrix

$$P_{11} := L_{11} - L_{12}(1 + L_{22})^{-1}L_{21} = L_{11} + L_{12}(1 + L_{22})^{-1}L_{12}^*$$

is nonnegative. Since the expression (1.6a) for K_{11} is equivalent to

$$K_{11} = P_{11}(1 + P_{11})^{-1},$$

it makes sense and, moreover, $0 \leq K_{11} < 1$. Likewise, the expression (1.6b) for K_{22} also makes sense and we have $0 \leq K_{22} < 1$. Consequently, the expressions (1.6c-d) for K_{12} and K_{21} also make sense. Thus, the transform $L \mapsto K$ is well-defined.

Now look at the relation (1.7a). It can be written as

$$L_{11} = Q_{11}(1 - Q_{11})^{-1},$$

where

$$Q_{11} = K_{11} + K_{12}(1 - K_{22})^{-1}K_{21}.$$

We have

$$1 - Q_{11} = 1 - K_{11} - K_{12}(1 - K_{22})^{-1}K_{21} = (1 - K_{11}) + K_{12}(1 - K_{22})^{-1}K_{12}^* > 0,$$

because $K_{11} < 1$. Since $L_{11} \geq 0$, we conclude $Q_{11} \geq 0$, which is the first inequality in (1.8). It is stronger than $K_{11} \geq 0$, because $Q_{11} \geq 0$ means

$$K_{11} - (K_{12}(1 - K_{22})^{-1}K_{12}^*) \geq 0.$$

Likewise, we establish the second inequality in (1.8).

Thus, the transform $L \mapsto K$ sends any matrix L satisfying (i) into a matrix K satisfying (ii). The inverse transform is justified in the same way. \square

Proposition 1.6 is a generalization of the trivial Proposition 1.3. Together with Proposition 1.4 it yields a class of finite point processes for which both the kernels L and K are given explicitly. We recall that L describes the distribution functions (see (1.1)), while K describes the correlation functions (see (1.2)).

Now we shall impose more special conditions on L : we shall assume that $L_{11} = 0$, $L_{22} = 0$. The meaning of this restriction is demonstrated by the following result.

Proposition 1.7. *Consider the process governed by a J -Hermitian kernel L with $L_{11} \geq 0$, $L_{22} \geq 0$. Then the condition $L_{11} = 0$, $L_{22} = 0$ exactly means that the process is concentrated on the configurations $\xi \subset \mathfrak{X}$ with the property*

$$|\xi \cap \mathfrak{X}_1| = |\xi \cap \mathfrak{X}_2|. \quad (1.9)$$

Proof. Assume $L_{11} = 0$, $L_{22} = 0$, and let ξ be a nonempty configuration. Its weight (1.1) is proportional to the value of the principal minor $\det L_\xi$. Remark that L_ξ is a block matrix whose diagonal blocks are zero. Such a matrix can be nondegenerate only if the blocks are of the same size, which means (1.9).

Conversely, let (1.9) hold. The diagonal blocks L_{11} , L_{22} are nonnegative, so that if one of them is nonzero, then it has a nonzero diagonal entry. That is, there exists a point $x \in \mathfrak{X}$ such that $L(x, x) > 0$. But then the one-point configuration $\xi = \{x\}$ has a nonzero weight which contradicts to the assumption (1.9). This contradiction implies that the diagonal blocks must be zero. \square

Proposition 1.8. *The transforms $L \mapsto K$ and $K \mapsto L$ define a bijective correspondence between*

(i) *the kernels L of the form*

$$L = \begin{bmatrix} 0 & A \\ -B & 0 \end{bmatrix}, \quad (1.10)$$

where the matrix $1 + AB$ is invertible (equivalently, $1 + BA$ is invertible) and

(ii) *the kernels K of the form*

$$K = \begin{bmatrix} CD & C \\ DCD - D & DC \end{bmatrix}, \quad (1.11)$$

where $1 - CD$ is invertible (equivalently, $1 - DC$ is invertible).

In terms of the blocks, this correspondence takes the form

$$C = (1 + AB)^{-1}A = A(1 + BA)^{-1}, \quad D = B, \quad (1.12a)$$

$$A = C(1 - DC)^{-1} = (1 - CD)^{-1}A, \quad B = D. \quad (1.12b)$$

In particular,

$$1 - CD = (1 + AB)^{-1}, \quad 1 - DC = (1 + BA)^{-1}. \quad (1.13)$$

Proof. This is a direct consequence of the formulas of Proposition 1.5 and the identity

$$X(1 \pm YX)^{-1} = (1 \pm XY)^{-1}X. \quad (1.14)$$

Indeed, take the kernel L of the form (1.10). Applying (1.6a-d) we get

$$K = \begin{bmatrix} AB(1 + AB)^{-1} & (1 + AB)^{-1}A \\ -(1 + BA)^{-1}B & BA(1 + BA)^{-1} \end{bmatrix}. \quad (1.15)$$

Using the identity (1.14) we verify that K has the desired form with C, D as indicated in (1.12a).

Conversely, starting with the kernel K of the form (1.11) and applying (1.7a-d) together with the identity (1.14) we readily verify that L has the form (1.10) with A, B indicated in (1.12b). \square

The following corollary will serve as a prompt for the main result of the next section (Theorem 2.4).

Corollary 1.9. *Let K be a J -Hermitian kernel of the form (1.11). Then*

$$L = \begin{bmatrix} 0 & D^* \\ -D & 0 \end{bmatrix}.$$

Proof. By Proposition 1.8, L is given by the formula (1.10) with $B = D$. Since K is J -Hermitian, L is J -Hermitian, too. This implies $A = B^* = D^*$. \square

Remark 1.10. Let us return to the very beginning of the section, where we introduced a class of point processes in \mathfrak{X} governed by kernels L with nonnegative principal minors (formula (1.1)). Let $\mathfrak{Y} \subseteq \mathfrak{X}$ be a subset. Given a point process in \mathfrak{X} we can define its ‘truncation’, which is a point process in \mathfrak{Y} : the latter process is the image of the former under the map $\xi \mapsto \xi \cap \mathfrak{Y}$. Clearly, the correlation functions of the truncated process are obtained simply by restricting the correlation functions of the initial process. So, if the initial process belongs to our class (i.e., is given by the formula (1.1)) then the truncated process also belongs to this class, and the ‘truncated’ kernel K is obtained simply by restricting the initial kernel K to $\mathfrak{Y} \times \mathfrak{Y}$. The corresponding transformation of the kernel L is more complicated. To describe it, write the kernel L in the block form with respect to the partition $\mathfrak{X} = \mathfrak{Y} \cup \overline{\mathfrak{Y}}$:

$$L = \begin{bmatrix} L_{\mathfrak{Y}\mathfrak{Y}} & L_{\mathfrak{Y}\overline{\mathfrak{Y}}} \\ L_{\overline{\mathfrak{Y}}\mathfrak{Y}} & L_{\overline{\mathfrak{Y}}\overline{\mathfrak{Y}}} \end{bmatrix}$$

Then the transformed kernel is equal to

$$L_{\mathfrak{Y}\mathfrak{Y}} - L_{\mathfrak{Y}\overline{\mathfrak{Y}}}(1 + L_{\overline{\mathfrak{Y}}\overline{\mathfrak{Y}}})^{-1}L_{\overline{\mathfrak{Y}}\mathfrak{Y}}.$$

This formula follows from (1.6a).

2. MAIN RESULT

Here we shall try to apply the general results of section 1 to the matrix Whittaker kernel introduced in [P.IV]. As we shall deal with continual kernels instead of finite matrices, we shall need to justify certain steps.

We take as \mathfrak{X} the punctured real line $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ equipped with Lebesgue measure dx . We write kernels on \mathbb{R}^* in the block form with respect to the partition

$$\mathbb{R}^* = \mathbb{R}_+ \cup \mathbb{R}_-.$$

We identify \mathbb{R}_- with \mathbb{R}_+ via the map $x \mapsto |x|$, $x < 0$, which enables us to interpret each block as a kernel on \mathbb{R}_+ and write

$$L^2(\mathbb{R}^*) = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+). \quad (2.1)$$

Given a kernel, we shall denote the corresponding integral operator by the same letter and we shall regard it as a Hilbert space operator.

We shall introduce the basic notation and then we shall recall the definition of the matrix Whittaker kernel from [P.IV].

This kernel depends on two parameters z, z' which play symmetric roles. The parameters satisfy the following restrictions:

$$\begin{aligned} &\text{either } z, z' \in \mathbb{C} \setminus \mathbb{Z} \text{ and } z' = \bar{z}, \\ &\text{or } z, z' \in \mathbb{R} \setminus \mathbb{Z} \text{ and } m < z, z' < m + 1 \text{ for a certain } m \in \mathbb{Z}. \end{aligned} \quad (2.2)$$

Instead of z, z' one can take the parameters a, μ defined by

$$a = \frac{z + z'}{2}, \quad \mu = \frac{z - z'}{2}. \quad (2.3)$$

Then the conditions (2.2) take the following form:

$$\begin{aligned} & a \text{ is always real and is not an integer when } \mu = 0; \\ & \mu \text{ is either pure imaginary or real; in the latter case} \\ & \text{there exists } m \in \mathbb{Z} \text{ such that } m + |\mu| < a < m + 1 - |\mu|. \end{aligned} \quad (2.4)$$

This implies that if μ is real then $|\mu| < \frac{1}{2}$.

The Whittaker function $W_{\kappa, \mu}(x)$, $x > 0$, with indices κ, μ , can be defined in terms of the confluent hypergeometric function ${}_1F_1$ as follows:

$$\begin{aligned} x^{-1/2} e^{x/2} W_{\kappa, \mu}(x) &= \frac{\Gamma(-2\mu)x^\mu}{\Gamma(\frac{1}{2} - \kappa - \mu)} {}_1F_1(\frac{1}{2} - \kappa + \mu; 2\mu + 1; x) \\ &+ \frac{\Gamma(2\mu)x^{-\mu}}{\Gamma(\frac{1}{2} - \kappa + \mu)} {}_1F_1(\frac{1}{2} - \kappa - \mu; -2\mu + 1; x). \end{aligned} \quad (2.5)$$

This expression makes sense when $\mu \neq 0$; when $\mu = 0$ (the so-called logarithmic case), it can be defined by a limit transition. We have

$$W_{\kappa, \mu} = W_{\kappa, -\mu}. \quad (2.6)$$

We shall always deal with a real κ and a real or pure imaginary μ ; then the Whittaker function takes real values.

The Whittaker function can be characterized as the only solution of the second order differential equation

$$\frac{d^2 W}{dx^2} + \left(-\frac{1}{4} + \frac{\kappa}{x} + \frac{\frac{1}{4} - \mu^2}{x^2} \right) W = 0, \quad x > 0, \quad (2.7)$$

which has exponential decay at $+\infty$:

$$W_{\kappa, \mu}(x) = x^\kappa e^{-x/2} \left(1 + O\left(\frac{1}{x}\right) \right), \quad x \rightarrow +\infty. \quad (2.8)$$

About the Whittaker function, see, e.g., [E1].

Let us fix z, z' (equivalently, a, μ) and abbreviate

$$\varphi(x) = x^{-1/2} e^{x/2} W_{a+\frac{1}{2}, \mu}(x), \quad (2.9a)$$

$$\varphi_-(x) = x^{-1/2} e^{x/2} W_{a-\frac{1}{2}, \mu}(x), \quad (2.9b)$$

$$\psi(x) = x^{-1/2} e^{x/2} W_{-a+\frac{1}{2}, \mu}(x), \quad (2.9c)$$

$$\psi_-(x) = x^{-1/2} e^{x/2} W_{-a-\frac{1}{2}, \mu}(x). \quad (2.9d)$$

Finally, let

$$\sigma = \sqrt{\sin(\pi z) \sin(\pi z')} = \sqrt{\frac{\cos(2\pi\mu) - \cos(2\pi a)}{2}} \quad (2.10)$$

and note that, under our restrictions on the parameters, σ^2 is always real and strictly positive; so, σ is always real and nonzero.

The *matrix Whittaker kernel*, as defined in [P.IV], has the form

$$[K] = \begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix}, \quad (2.11)$$

where the blocks are the following kernels on \mathbb{R}_+ :

$$K_{++}(x, y) = \frac{1}{\Gamma(z)\Gamma(z')} \frac{\varphi(x)\varphi_-(y) - \varphi_-(x)\varphi(y)}{x - y} \quad (2.12a)$$

$$K_{--}(x, y) = \frac{1}{\Gamma(-z)\Gamma(-z')} \frac{\psi(x)\psi_-(y) - \psi_-(x)\psi(y)}{x - y} \quad (2.12b)$$

$$K_{+-}(x, y) = \frac{\sigma}{\pi} \frac{\varphi(x)\psi(y) + zz'\varphi_-(x)\psi_-(y)}{x + y} \quad (2.12c)$$

$$K_{-+}(x, y) = -K_{+-}(y, x) \quad (2.12d)$$

Note that all the blocks are real kernels and the diagonal kernels K_{++} , K_{--} are symmetric; together with the last relation this implies that $[K]$ is J -symmetric.

Proposition 2.1. *The matrix Whittaker kernel $[K]$ can be written in the form*

$$[K] = \begin{bmatrix} CD & C \\ DCD - D & DC \end{bmatrix}, \quad (2.13)$$

where

$$C(x, y) = K_{+-}(x, y), \quad (2.14)$$

$$D(x, y) = \frac{\sigma}{\pi} \left(\frac{x}{y} \right)^{-a} \frac{e^{-\frac{x+y}{2}}}{x + y}, \quad (2.15)$$

and, by definition, the products CD , DC , DCD are superpositions of integral operators:

$$\begin{aligned} (CD)(x, y) &= \int_0^{+\infty} C(x, s)D(s, y)ds \\ (DC)(x, y) &= \int_0^{+\infty} D(x, s)C(s, y)ds \\ (DCD)(x, y) &= \int_0^{+\infty} \int_0^{+\infty} D(x, s_1)C(s_1, s_2)D(s_2, y)ds_1ds_2. \end{aligned} \quad (2.16)$$

Proof. This is merely a reformulation of the results of [P.IV], section 2. Indeed, let

$$N(x, y) = \frac{\sigma}{\pi} \left(\frac{x}{y} \right)^a e^{-\frac{x+y}{2}} K_{+-}(x, y), \quad W(x, y) = \frac{1}{x + y}, \quad (2.17)$$

and let R_1 and R_2 be the following multiplication operators:

$$(R_1 f)(x) = \pi \sigma^{-1} x^{-a} e^{x/2} f(x), \quad (R_2 f)(x) = x^{-a} e^{-x/2} f(x). \quad (2.18)$$

In [P.IV], section 2, it was proved that

$$K = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \cdot \begin{bmatrix} NW & N \\ WNW - W & WN \end{bmatrix} \cdot \begin{bmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{bmatrix}, \quad (2.19)$$

which means, in particular, that all the integrals involved in (2.16) make sense. It follows that $[K]$ has the form (2.13) with

$$C = R_1 N R_2^{-1}, \quad D = R_2 W R_1^{-1},$$

which are exactly (2.14), (2.15). \square

Proposition 2.2. *Assume $-1/2 < a < 1/2$. Then the integral operator D with the kernel $D(x, y)$ as defined in (2.15) is bounded in $L^2(\mathbb{R}_+, dx)$.*

Proof. Set

$$D'(x, y) = \left(\frac{x}{y}\right)^{-a} \frac{1}{x+y}. \quad (2.20)$$

We have $D = (\sigma/\pi) M D' M$, where M stands for the operator of multiplication by the bounded function $e^{-x/2}$, so that it suffices to check that D' is bounded.

Passing to new variables ξ, η such that $x = e^{-\xi}$, $y = e^{-\eta}$, we transform D' to the integral operator in $L^2(\mathbb{R}, d\xi)$ with the kernel

$$\mathcal{D}'(\xi, \eta) = \frac{e^{-a(\xi-\eta)}}{e^{(\xi-\eta)/2} + e^{-(\xi-\eta)/2}}. \quad (2.21)$$

Since this kernel is translation invariant, the Fourier transform takes \mathcal{D}' to a multiplication operator. It remains to check that the latter is bounded.

To this end we employ the formula

$$\text{Fourier}_u \left\{ \frac{e^{\alpha v}}{e^{\beta v} + e^{-\beta v}} \right\} = \frac{\pi}{2\beta} \frac{1}{\cos(\pi(-\frac{\alpha}{2\beta} + \frac{iu}{2\beta}))} \quad (2.22)$$

$$\beta > 0, \quad \Re \alpha < \beta$$

where

$$\text{Fourier}_u \{f(v)\} = g(u) \quad \text{means} \quad g(u) = \int e^{iuv} f(v) dv,$$

see [E2, 3.2(15)].

Applying (2.22) we get that after the Fourier transform the operator \mathcal{D}' becomes the operator of multiplication by the function

$$\frac{\pi}{\cos(\pi(a + iu))},$$

which is bounded by the assumption $|a| < 1/2$. \square

Proposition 2.3. *Assume $-1/2 < a < 1/2$. Then the integral operator D with the kernel $C(x, y) = K_{+-}(x, y)$ as defined by the formulas (2.14), (2.12c), (2.10), (2.9), (2.5), is bounded in $L^2(\mathbb{R}_+, dx)$.*

Proof. Assume first that $\mu \neq 0$. Then, according to (2.5), we can write

$$x^{-1/2}W_{\kappa, \mu}(x) = x^\mu(\dots) + x^{-\mu}(\dots), \quad (2.23)$$

where each of the two expressions denoted by dots is equal to a confluent hypergeometric function multiplied by an exponential factor and so is analytic on the whole real axis. Consequently, the kernel $C(x, y)$ can be represented in the form

$$C(x, y) = \frac{x^\mu y^\mu a_{++}(x, y) + x^{-\mu} y^{-\mu} a_{--}(x, y) + x^\mu y^{-\mu} a_{+-}(x, y) + x^{-\mu} y^\mu a_{-+}(x, y)}{x + y},$$

where the four functions a_{++} , a_{--} , a_{+-} , a_{-+} are analytic about $(0, 0)$.

The key observation is that

$$a_{++}(0, 0) = a_{--}(0, 0) = 0.$$

Indeed, it follows from (2.5) that

$$\begin{aligned} a_{++}(0, 0) &= \frac{\sigma}{\pi} (\Gamma(-2\mu))^2 \left(\frac{1}{\Gamma(-z)\Gamma(z')} + \frac{zz'}{\Gamma(-z+1)\Gamma(z'+1)} \right) \\ a_{--}(0, 0) &= \frac{\sigma}{\pi} (\Gamma(-2\mu))^2 \left(\frac{1}{\Gamma(-z')\Gamma(z)} + \frac{zz'}{\Gamma(-z'+1)\Gamma(z+1)} \right), \end{aligned}$$

and the both expressions in the parentheses are equal to zero.

Consider the kernel

$$C_0(x, y) = \frac{x^\mu y^{-\mu} a_{+-}(0, 0) + x^{-\mu} y^\mu a_{-+}(0, 0)}{x + y},$$

and set

$$C'(x, y) = C(x, y) - \chi_{[0,1]}(x)C_0(x, y)\chi_{[0,1]}(y),$$

where $\chi_{[0,1]}$ stands for the characteristic function of $[0, 1]$. The boundedness of the integral operator C will follow from the two claims:

- *Claim 1.* The kernel $C_0(x, y)$ defines a bounded operator in $L^2(\mathbb{R}_+, dx)$.
- *Claim 2.* The function $C'(x, y)$ is square integrable on \mathbb{R}_+^2 .

Indeed, $C(x, y)$ is the sum of $\chi_{[0,1]}(x)C_0(x, y)\chi_{[0,1]}(y)$ and $C'(x, y)$. Since the function $\chi_{[0,1]}$ is bounded, the first summand defines a bounded operator by Claim 1, and the second summand defines a Hilbert–Schmidt (hence bounded) operator by Claim 2.

Let us check Claim 1. It suffices to show that the kernel

$$\left(\frac{x}{y}\right)^{\pm\mu} \frac{1}{x + y}$$

defines a bounded operator. Arguing as in the proof of Proposition 2.2 we reduce this to the boundedness of the function

$$\frac{\pi}{\cos(\mp\pi\mu + i\pi u)}.$$

Since $\mu \in i\mathbb{R}$ or $-1/2 < \mu < 1/2$ this is obvious.

Let us check Claim 2. It suffices to show that C' is square integrable both in the square $[0, 1]^2$ and in its complement $\mathbb{R}_+^2 \setminus [0, 1]^2$.

In the second region, C' coincides with C . Recall that $C(x, y) = K_{+-}(x, y)$ (see (2.14)) and look at the expression (2.12c) for $K_{+-}(x, y)$. Outside $[0, 1]^2$, the fraction $\frac{1}{x+y}$ is bounded, so that it remains to prove the square integrability of the numerator of (2.12c). To this end it suffices to prove that each of the functions $\varphi(x)\psi(y)$, $\varphi_-(x)\psi_-(y)$ is square integrable in our region. Actually we can claim that they are square integrable in the whole quadrant \mathbb{R}_+^2 . Indeed, this reduces to the fact that each of the four functions φ , ψ , φ_- , ψ_- is square integrable in \mathbb{R}_+ . Each of these functions is of the form $x^{-1/2}W_{\kappa,\mu}(x)$. The latter function has exponential decay at infinity, hence it is square integrable near infinity. Finally, near zero, it behaves as indicated in (2.23). Since μ is either pure imaginary or satisfies $-1/2 < \mu < 1/2$ we get square integrability about zero as well.

Let us examine the kernel $C'(x, y)$ in the square $[0, 1]^2$; here it coincides with $C(x, y) - C_0(x, y)$. By the definition of C_0 we have

$$C'(x, y) = \frac{x^\mu y^\mu b_{++}(x, y) + x^{-\mu} y^{-\mu} b_{--}(x, y) + x^\mu y^{-\mu} b_{+-}(x, y) + x^{-\mu} y^\mu b_{-+}(x, y)}{x + y},$$

where the functions

$$\begin{aligned} b_{++}(x, y) &= a_{++}(x, y), & b_{--}(x, y) &= a_{--}(x, y), \\ b_{+-}(x, y) &= a_{+-}(x, y) - a_{+-}(0, 0), & b_{-+}(x, y) &= a_{-+}(x, y) - a_{-+}(0, 0) \end{aligned}$$

vanish at $(0, 0)$. Each of these four functions can be written in the form $x(\dots) + y(\dots)$ where the expressions denoted by dots are certain analytic (hence bounded) functions. Since the expressions $\frac{x}{x+y}$ and $\frac{y}{x+y}$ are bounded, we have to examine the functions of the form $x^\varepsilon y^\delta$ where ε, δ take the values $\pm\mu$. Since μ is pure imaginary or $|\mu| < 1/2$, the latter functions are square integrable. This completes the proof of Claim 2.

Thus, we have proved the proposition for the case $\mu \neq 0$. In the logarithmic case $\mu = 0$ the argument is quite similar; we shall only indicate necessary modifications.

We have from (2.5)

$$x^{-1/2}W_{\kappa,0} = \lim_{\mu \rightarrow 0} x^{-1/2}W_{\kappa,\mu}(x) = \ln x \cdot a_0(x) + a_1(x), \quad (2.24)$$

where a_0 and a_1 are certain analytic functions such that

$$\begin{aligned} a_0(0) &= -\frac{1}{\Gamma(\frac{1}{2} - \kappa)} \\ a_1(0) &= -\frac{1}{\Gamma(\frac{1}{2} - \kappa)}(\psi(\frac{1}{2} - \kappa) - 2\psi(1)), \quad \psi(\cdot) := \frac{\Gamma'(\cdot)}{\Gamma(\cdot)} \end{aligned} \quad (2.25)$$

see formulas (6.9(2)) and (6.7(13)) in [E1]. From this we get

$$C(x, y) = \frac{\ln x \ln y \cdot a_{00}(x, y) + \ln x \cdot a_{01}(x, y) + \ln y \cdot a_{10}(x, y) + a_{11}(x, y)}{x + y},$$

where a_{00} , a_{01} , a_{10} , a_{11} are certain analytic functions. Further, using (2.25) and the well-known relation $\psi(1+a) - \psi(a) = a^{-1}$ we get

$$a_{00}(0, 0) = 0, \quad a_{01}(0, 0) = -a_{10}(0, 0).$$

Now we set

$$C_0(x, y) = \frac{\ln x \cdot a_{01}(0, 0) + \ln y \cdot a_{10}(0, 0)}{x + y} = a_{01}(0, 0) \frac{\ln\left(\frac{x}{y}\right)}{x + y},$$

we define $C'(x, y)$ as above and we state the same two claims as above, which imply the proposition.

To check Claim 1 we again pass to new variables and transform the kernel $C(x, y)$ to

$$\text{const} \frac{\xi - \eta}{e^{(\xi-\eta)/2} + e^{-(\xi-\eta)/2}}. \quad (2.26)$$

Employing the formula

$$\text{Fourier}_u \left\{ \frac{v}{e^{v/2} + e^{-v/2}} \right\} = \pi^2 \frac{\text{sh}(\pi u)}{\text{ch}^2(\pi u)} \quad (2.27)$$

we get that after the Fourier transform the kernel (2.26) becomes multiplication by the function (2.27) times a scalar factor. Then we remark that the latter function is bounded.

As for Claim 2, it is verified in exactly the same way as in the nonlogarithmic case: here we employ the fact that the function (2.24) has exponential decay at infinity and is square integrable near zero.

This completes the proof. \square

Theorem 2.4. *Consider the operator K in the Hilbert space $L^2(\mathbb{R}_+, dx) \oplus L^2(\mathbb{R}_+, dx)$ defined by the matrix Whittaker kernel (2.11)–(2.12), and assume that the parameter $a = \frac{z+z'}{2}$ satisfies the condition $-1/2 < a < 1/2$. Set*

$$L = \begin{bmatrix} 0 & A \\ -B & 0 \end{bmatrix}, \quad (2.28)$$

where

$$A(x, y) = D(y, x) = \frac{\sigma}{\pi} \left(\frac{x}{y} \right)^{-a} \frac{e^{-\frac{x+y}{2}}}{x + y} \quad (2.29)$$

$$B(x, y) = D(x, y) = A(y, x) = \frac{\sigma}{\pi} \left(\frac{x}{y} \right)^a \frac{e^{-\frac{x+y}{2}}}{x + y}. \quad (2.30)$$

(Note that L is bounded because D is bounded by Proposition 2.2).

Then we have

$$K = \frac{L}{1+L}. \quad (2.31)$$

Proof. We know from Propositions 2.2, 2.3 that the kernels $C(x, y)$, $D(x, y)$ (see (2.14), (2.15)) define bounded operators C , D . Together with Proposition 2.1 this means that in the formula (2.13) for the matrix Whittaker kernel we may interpret all the products as products of bounded operators. In particular, this implies that K is bounded.

Furthermore, the fact that the matrix Whittaker kernel is J -Hermitian implies the following operator relation:

$$K = \begin{bmatrix} CD & C \\ DCD - D & DC \end{bmatrix} = \begin{bmatrix} D^*C^* & D^* - D^*C^*D^* \\ -C^* & C^*D^* \end{bmatrix}. \quad (2.32)$$

(Here the adjoint operators $(\dots)^*$ coincide with the transposed ones because all the operators are real.)

The desired relation (2.13) is equivalent to $K + KL = L$, which in turn means that

$$\begin{bmatrix} CD & C \\ DCD - D & DC \end{bmatrix} + \begin{bmatrix} CD & C \\ DCD - D & DC \end{bmatrix} \begin{bmatrix} 0 & D^* \\ -D & 0 \end{bmatrix} = \begin{bmatrix} 0 & D^* \\ -D & 0 \end{bmatrix},$$

or

$$\begin{bmatrix} 0 & C + CDD^* \\ -D & DC + DCDD^* - DD^* \end{bmatrix} = \begin{bmatrix} 0 & D^* \\ -D & 0 \end{bmatrix},$$

or

$$C + CDD^* = D^*, \quad DC + DCDD^* - DD^* = 0.$$

Now, the latter two relations are direct consequences of (2.32). \square

The result seems to be quite surprising. First, the expression for the kernel L is very simple and involves no special functions. Second, let us pass from the parameters z, z' to the parameters a, μ ; then we see that in (2.28)–(2.30), μ occurs only in the scalar factor σ .

Corollary 2.5. *Let a be fixed, $-1/2 < a < 1/2$. Then the operators K corresponding to various values of the parameter μ pairwise commute. The same holds for the blocks K_{++} or the blocks K_{--} .*

Proof. According to (2.29), (2.30) we can write

$$A = \sigma A_0, \quad B = \sigma B_0$$

where the operators A_0 and B_0 do not depend on μ . Consequently,

$$L = \sigma \begin{bmatrix} 0 & A_0 \\ -B_0 & 0 \end{bmatrix}$$

This means that when μ varies, the operators L differ by a scalar fact only. Since $K = L(1+L)^{-1}$, we conclude that the corresponding operators K form a commutative family.

Next, by (1.15),

$$K_{++} = AB(1+AB)^{-1}. \quad (2.33)$$

Since $AB = \sigma^2 A_0 B_0$ where $A_0 B_0$ does not depend on μ , the operators AB with various μ form a commutative family. So, the same holds for the operators K_{++} .

For the blocks K_{--} the argument is the same. \square

This suggests the idea to take as the parameters the couple a, σ .

3. DIAGONALIZATION OF THE KERNELS

Now we are in a position to perform the spectral analysis of the kernel L — to “diagonalize” it in a continual basis and hence to “diagonalize” the kernel K , too.

Proposition 3.1. *The “ordinary” Whittaker kernel K_{++} with parameters a, μ commutes with the Sturm–Liouville differential operator*

$$\mathfrak{D}(a) = -\frac{d}{dx} x^2 \frac{d}{dx} + \left(a - \frac{x}{2}\right)^2, \quad (3.1)$$

i.e., the kernel satisfies the following differential equation

$$\mathfrak{D}(a)_x K_{++}(x, y) = \mathfrak{D}(a)_y K_{++}(x, y), \quad (3.2)$$

where the subscript x or y indicates the variable on which the differential operator acts.

Proof. This is a limit case of Proposition 6.2 in [P.III] and can be verified by a direct computation. We do not give a detailed proof, because we shall employ this result as a prompt only. \square

It is worth noting that $\mathfrak{D}(a)$ does not depend on μ , which agrees with the fact that the kernels K_{++} with varying μ form a commutative family (Corollary 2.5).

Consider the following functions on \mathbb{R}_+ :

$$f_{a,m}(x) = \frac{1}{x} W_{a,im}(x), \quad m > 0. \quad (3.3)$$

We have

$$\mathfrak{D}(a)f_{a,m} = (a^2 + \frac{1}{4} + m^2)f_{a,m}. \quad (3.4)$$

According to [W], the functions $f_{a,m}$ with a fixed and m ranging over \mathbb{R}_+ form a continual basis in $L^2(\mathbb{R}_+)$ diagonalizing $\mathfrak{D}(a)$. Moreover, an explicit Plancherel formula holds:

$$(f, g)_{L^2(\mathbb{R}_+)} = \int_0^{+\infty} \frac{(f, f_{a,m})(f_{a,m}, g)}{(f_{a,m}, f_{a,m})} dm, \quad (3.5)$$

where

$$(f_{a,m}, f_{a,m}) := \frac{\pi^2}{\Gamma(\frac{1}{2} - a - im)\Gamma(\frac{1}{2} - a + im)}. \quad (3.6)$$

Consider the decomposition (2.1) and take in its first component the basis $\{f_{a,m}\}_{m \geq 0}$ and in its second component — the basis $\{f_{-a,m}\}_{m \geq 0}$. Together they form a (continual) basis in the whole space $L^2(\mathbb{R}^*)$. The following claim describes the diagonalization of L in this basis.

Theorem 3.2. *Let A and B be as in (2.29), (2.30), and assume that $|a| < \frac{1}{2}$. Then we have*

$$Af_{-a,m} = \frac{\sigma}{\pi} \Gamma(\frac{1}{2} - a + im)\Gamma(\frac{1}{2} - a - im)f_{a,m} \quad (3.7a)$$

$$Bf_{a,m} = \frac{\sigma}{\pi} \Gamma(\frac{1}{2} + a + im)\Gamma(\frac{1}{2} + a - im)f_{-a,m}. \quad (3.7b)$$

Proof. The function $Af_{-a,m}$ is essentially the Stieltjes transform of the function

$$y^{-a-1} \exp(-\frac{1}{2}y) W_{-a,im}(y),$$

which is given, under the assumption $\Re(-a) > -\frac{1}{2}$, in [E2, 14.3(53)]. The expression for $B = A^*$ is obtained in the same way; here we need $\Re(a) > -\frac{1}{2}$. Thus, we have entirely used the assumption $|a| < \frac{1}{2}$. \square

Remark 3.3. We have

$$ABf_{a,m} = \frac{\cos(2\pi\mu) - \cos(2\pi a)}{\cos(2\pi im) + \cos(2\pi a)} f_{a,m}. \quad (3.8)$$

This shows that $a = \pm\frac{1}{2}$ are “critical” points: when $|a| < \frac{1}{2}$, the operator $AB = AA^*$ is bounded as its spectrum is bounded, whence A and B are bounded. But in the limit $a \rightarrow \pm\frac{1}{2}$ the spectrum of AB becomes unbounded.

Employing (2.33), we get, as a corollary of (2.18), a diagonalization of K_{++} :

Corollary 3.4. Assume $|a| < \frac{1}{2}$. We have

$$K_{++}f_{a,m} = \frac{\cos(2\pi\mu) - \cos(2\pi a)}{\cos(2\pi\mu) + \cos(2\pi im)} f_{a,m}.$$

Remark 3.5. The operator A for $a = 0$ arose before in the asymptotic analysis of the Painlevé transcendent of the third kind [MTW] and two-dimensional Ising model [T]. It has also been used in approximating the resolvent of a certain more difficult operator in [TW4].

The spectral analysis of the kernel (2.29) for $a = 0$ was carried out in [MTW], see also [T].

If $a = 0$ then (3.3), (3.7a) turns into

$$f_{0,m} = \frac{1}{x} W_{0,im}(x) = \frac{1}{\sqrt{\pi x}} K_{im}\left(\frac{x}{2}\right),$$

$$Af_{0,m} = \frac{\sigma}{\operatorname{ch} \pi m} f_{0,m}$$

which agrees with the results of [MTW] (here $K_\nu(x)$ stands for the Bessel K -function).

4. THE MATRIX TAIL KERNEL

In [P.III], we studied a ‘tail kernel’ associated to the Whittaker kernel. This is a translation invariant kernel on \mathbb{R} generalizing the sine kernel. Specifically, it has the form

$$\frac{B}{A} \frac{\sin(A(\xi - \eta))}{\operatorname{sh}(B(\xi - \eta))} \quad \text{or} \quad \frac{B \operatorname{sh}(A(\xi - \eta))}{A \operatorname{sh}(B(\xi - \eta))} \quad (4.1)$$

depending on whether μ is pure imaginary or real, respectively; A and B are certain constants depending on z, z' (see also below). Here we aim to study a similar object for the *matrix* Whittaker kernel.

The proofs are omitted; they are quite similar to that given in [P.III]. For the sake of simplicity, we shall assume $z \neq z'$.

Let us briefly recall how the tail kernel arises. According to [P.II], Theorem 4.1.1, the Whittaker kernel $K_{++}(x, y)$ behaves near $(0, 0)$ as follows

$$K_{++}(x, y) \approx \frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi(z - z'))} \frac{1}{\sqrt{xy}} \frac{(x/y)^{\frac{z-z'}{2}} - (x/y)^{-\frac{z-z'}{2}}}{(x/y)^{\frac{1}{2}} - (x/y)^{-\frac{1}{2}}} \quad (4.2)$$

In particular, on the diagonal,

$$K_{++}(x, x) \approx \frac{C}{x},$$

where

$$C = C(z, z') = \frac{(z - z') \sin(\pi z) \sin(\pi z')}{\pi \sin(\pi(z - z'))}. \quad (4.3)$$

We pass to new variables to make the density function $K_{++}(x, x)$ asymptotically equal to 1. Specifically, we take

$$x = e^{-\xi/C}, \quad y = e^{-\eta/C}.$$

Then the resulting kernel in ξ, η takes the form

$$\mathcal{K}_{++}(\xi, \eta) + \text{remainder term}$$

where the remainder term tends to zero as $\xi, \eta \rightarrow +\infty$ and \mathcal{K}_{++} is a translation invariant kernel equal to 1 on the diagonal,

$$\mathcal{K}_{++}(\xi, \eta) = \frac{1}{z - z'} \frac{\text{sh}(A(\xi - \eta))}{\text{sh}(B(\xi - \eta))} \quad (4.4)$$

with

$$B = \frac{1}{2C} = \frac{\pi \sin(\pi(z - z'))}{2(z - z') \sin(\pi z) \sin(\pi z')} \quad (4.5)$$

$$A = (z - z')B = \frac{\pi \sin(\pi(z - z'))}{2 \sin(\pi z) \sin(\pi z')}. \quad (4.6)$$

Note that B is real and strictly positive while A is real or pure imaginary together with μ (so, for pure imaginary μ the kernel actually has the form of the first expression in (4.1)).

Theorem 4.1. *Application of the same procedure to the matrix Whittaker kernel*

$$K(x, y) = \begin{bmatrix} K_{++}(x, y) & K_{+-}(x, y) \\ K_{-+}(x, y) & K_{--}(x, y) \end{bmatrix} \quad (4.7)$$

leads to a translation invariant block kernel in ξ, η ,

$$\mathcal{K}(\xi, \eta) = \begin{bmatrix} \mathcal{K}_{++}(\xi, \eta) & \mathcal{K}_{+-}(\xi, \eta) \\ \mathcal{K}_{-+}(\xi, \eta) & \mathcal{K}_{--}(\xi, \eta) \end{bmatrix}, \quad (4.8)$$

where all the blocks are real,

$$\mathcal{K}_{++}(\xi, \eta) = \mathcal{K}_{--}(\xi, \eta), \quad \mathcal{K}_{+-}(\xi, \eta) = -\mathcal{K}_{-+}(\eta, \xi), \quad (4.9)$$

\mathcal{K}_{++} is as in (4.4) and

$$\mathcal{K}_{+-}(\xi, \eta) = \frac{1}{\sqrt{\sin(\pi z) \sin(\pi z')}} \frac{1}{z - z'} \frac{\sin(\pi z) e^{A(\xi - \eta)} - \sin(\pi z') e^{-A(\xi - \eta)}}{e^{B(\xi - \eta)} + e^{-B(\xi - \eta)}} \quad (4.10)$$

with A and B as in (4.5), (4.6).

I.e., in the new variables, the kernel (4.7) takes the form (4.8) plus a remainder term which tends to zero as $\xi, \eta \rightarrow +\infty$.

Idea of proof. This claim is a generalization of Theorem 3.2 from [P.III] and is proved in the same way. In addition to (4.2) we employ an asymptotic formula for the kernel $K_{+-}(x, y)$ near $(0, 0)$:

$$K_{+-}(x, y) \approx \frac{\sqrt{\sin(\pi z) \sin(\pi z')}}{\pi \sin(\pi(z - z'))} \frac{1}{\sqrt{xy}} \frac{\sin(\pi z)(x/y)^{\frac{z-z'}{2}} + \sin(\pi z')(x/y)^{-\frac{z-z'}{2}}}{(x/y)^{\frac{1}{2}} + (x/y)^{-\frac{1}{2}}}, \quad (4.11)$$

which is proved similarly. \square

Consider the integral operator in the Hilbert space of square integrable \mathbb{C}^2 -valued functions on \mathbb{R} that is defined by the kernel $\mathcal{K}(\xi, \eta)$. Since the kernel is translation invariant, the operator in question is a convolution operator. Under the Fourier transform it turns into the operator of multiplication by a 2×2 matrix-valued function, say, $\widehat{\mathcal{K}}(u)$.

Proposition 4.2. *The above defined matrix function on \mathbb{R} has the form*

$$\widehat{\mathcal{K}}(u) = \begin{bmatrix} f(u) & g(u) \\ -\bar{g}(u) & f(u) \end{bmatrix}, \quad (4.12)$$

where $f(u)$ is a real function, $g(u)$ is a complex function, $\bar{g}(u) = \overline{g(u)}$,

$$\begin{aligned} f(u) &= 2 \sin(\pi z) \sin(\pi z') \frac{1}{\cos(\pi(z - z')) + \operatorname{ch}(\pi u/B)} \\ g(u) &= 2 \sqrt{\sin(\pi z) \sin(\pi z')} \frac{\cos(\pi(z + z')/2 + i\pi u/(2B))}{\cos(\pi(z - z')) + \operatorname{ch}(\pi u/B)}. \end{aligned}$$

Sketch of proof. Rewrite the expressions (4.4), (4.10) in the form

$$\mathcal{K}_{++}(\xi, \eta) = k_{++}(\xi - \eta), \quad \mathcal{K}_{+-}(\xi, \eta) = k_{+-}(\xi - \eta)$$

where k_{++} and k_{+-} are functions of a single variable, say, ζ . It follows from the symmetry properties (4.9) that $\widehat{\mathcal{K}}(u)$ has the form (4.12), where f and g are the Fourier transforms of k_{++} and k_{+-} , respectively:

$$f(u) = \int e^{i\pi u \zeta} k_{++}(\zeta) d\zeta, \quad g(u) = \int e^{i\pi u \zeta} k_{+-}(\zeta) d\zeta.$$

Since k_{++} is an even function, $f(u)$ is real.

The desired explicit expression for $f(u)$ is a table integral, see, e.g., [E2, 1.9(14)]. The explicit expression for $g(u)$ can be derived from another table integral, see [E2, 3.2(15)]. \square

5. DEGENERATION TO A BESSEL-TYPE KERNEL

We shall need three Bessel functions: the Bessel function of the first kind

$$J_\nu(X) = \frac{(X/2)^\nu}{\Gamma(\nu+1)} {}_0F_1(\nu+1; -(X/2)^2) = \frac{(X/2)^\nu}{\Gamma(\nu+1)} \sum_{m \geq 0} \frac{(-1)^m (X/2)^{2m}}{m!(\nu+1)_m},$$

the modified Bessel function of the first kind

$$I_\nu(X) = \frac{(X/2)^\nu}{\Gamma(\nu+1)} {}_0F_1(\nu+1; (X/2)^2) = \frac{(X/2)^\nu}{\Gamma(\nu+1)} \sum_{m \geq 0} \frac{(X/2)^{2m}}{m!(\nu+1)_m},$$

and the modified Bessel function of the third kind, also called the Macdonald function

$$K_\nu(X) = \frac{\pi}{2 \sin(\pi\nu)} (I_{-\nu}(X) - I_\nu(X)).$$

Here we assume $X > 0$.

We fix the parameters z_0, z'_0 satisfying the assumptions (2.2), and we set

$$a_0 = \frac{z_0 + z'_0}{2}, \quad \mu = \frac{z_0 - z'_0}{2}.$$

In terms of a_0, μ , the restrictions on z_0, z'_0 take the form (2.4).

Let N be an integer; then the parameters

$$z = z_0 + N, \quad z' = z'_0 + N$$

will also satisfy the same restrictions (2.2) as z_0, z'_0 . We set

$$a := \frac{z + z'}{2}$$

and note that

$$a = a_0 + N, \quad \frac{z - z'}{2} = \mu.$$

We introduce four functions in a positive variable ξ :

$$\begin{aligned} A(\xi) &= \frac{\sin(\pi z_0) J_{2\mu}(2\sqrt{\xi}) - \sin(\pi z'_0) J_{-2\mu}(2\sqrt{\xi})}{\sin(2\pi\mu)}, \\ B(\xi) &= K_{2\mu}(2\sqrt{\xi}), \\ \tilde{A}(\xi) &= \sqrt{\xi} \frac{\sin(\pi z_0) J'_{2\mu}(2\sqrt{\xi}) - \sin(\pi z'_0) J'_{-2\mu}(2\sqrt{\xi})}{\sin(2\pi\mu)}, \\ \tilde{B}(\xi) &= \sqrt{\xi} K'_{2\mu}(2\sqrt{\xi}), \end{aligned}$$

where

$$J'_\nu(X) = \frac{d}{dX} J_\nu(X), \quad K'_\nu(X) = \frac{d}{dX} K_\nu(X).$$

Note that

$$\tilde{A}(\xi) = \left(\xi \frac{d}{d\xi} \right) A(\xi), \quad \tilde{B}(\xi) = \left(\xi \frac{d}{d\xi} \right) B(\xi),$$

because, for a function $f(\cdot)$,

$$\left(\xi \frac{d}{d\xi} \right) f(2\sqrt{\xi}) = \sqrt{\xi} f'(2\sqrt{\xi}).$$

Finally, we let $N \rightarrow \infty$ and associate with the positive variables x, y the ‘scaled variables’ ξ, η ,

$$x = \xi/N, \quad y = \eta/N.$$

Theorem 5.1. *In the scaled limit, as $N \rightarrow +\infty$ inside $2\mathbb{Z}$, the matrix Whittaker kernel in the variables x, y ,*

$$K(x, y) = \begin{bmatrix} K_{++}(x, y) & K_{+-}(x, y) \\ K_{-+}(x, y) & K_{--}(x, y) \end{bmatrix}$$

tends to a matrix kernel K^{lim} in the variables ξ, η with the following blocks:

$$K_{++}^{lim}(\xi, \eta) = \frac{A(\xi)\tilde{A}(\eta) - \tilde{A}(\xi)A(\eta)}{\xi - \eta}, \quad (5.1a)$$

$$K_{+-}^{lim}(\xi, \eta) = -\frac{2\sqrt{\sin(\pi z_0)\sin(\pi z'_0)}}{\pi} \frac{A(\xi)\tilde{B}(\eta) - \tilde{A}(\xi)B(\eta)}{\xi + \eta}, \quad (5.1b)$$

$$K_{-+}^{lim}(\xi, \eta) = -K_{+-}^{lim}(\eta, \xi), \quad (5.1c)$$

$$K_{--}^{lim}(\xi, \eta) = \frac{4\sin(\pi z_0)\sin(\pi z'_0)}{\pi^2} \frac{B(\xi)\tilde{B}(\eta) - \tilde{B}(\xi)B(\eta)}{\xi - \eta}. \quad (5.1d)$$

Comments. 1) Under the shift $z_0 \mapsto z_0 + 1$, $z'_0 \mapsto z'_0 + 1$, the functions B, \tilde{B} remain stable while the functions A, \tilde{A} are multiplied by -1 . It follows that under this shift, the diagonal blocks $K_{++}^{lim}, K_{--}^{lim}$ are stable while the blocks $K_{+-}^{lim}, K_{-+}^{lim}$ are multiplied by -1 . We could assume N tends to infinity inside \mathbb{Z} (instead of $2\mathbb{Z}$) by introducing in the blocks K_{+-}, K_{-+} the extra factor $(-1)^N$; such a factor does not affect the correlation functions.

2) Note that the function B depends only on the parameter μ while A depends on the both parameters a_0, μ . This results in a strong asymmetry between the diagonal blocks K_{++}^{lim} and K_{--}^{lim} . Of course, these blocks are interchanged if we let N tend to $-\infty$ instead of $+\infty$.

3) When μ is real and one of the parameters $z_0 = a_0 + \mu$, $z'_0 = a_0 - \mu$ becomes integer, the kernel K_{++} degenerates to the conventional Bessel kernel²

$$\frac{J_\nu(2\sqrt{\xi})\sqrt{\eta}J'_\nu(2\sqrt{\eta}) - \sqrt{\xi}J'_\nu(2\sqrt{\xi})J_\nu(2\sqrt{\eta})}{\xi - \eta}, \quad \nu = 2\mu. \quad (5.2)$$

This agrees with the degeneration of the Whittaker kernel to the Laguerre kernel, see [P.III], Remark 2.4.³ Thus, thanks to the parameter a_0 , the kernel K_{++}^{lim} provides a deformation of the Bessel kernel; one more new point is that the index μ in the expression for K_{++}^{lim} can be pure imaginary.

4) The kernel

$$K_{--}^{lim}(\xi, \eta) = \text{const} \frac{K_{2\mu}(2\sqrt{\xi})\sqrt{\eta}K'_{2\mu}(2\sqrt{\eta}) - \sqrt{\xi}K'_{2\mu}(2\sqrt{\xi})K_{2\mu}(2\sqrt{\eta})}{\xi - \eta} \quad (5.3)$$

except the scalar factor *const*, depends only on μ and looks quite similar to the Bessel kernel (5.2). One could call it the *Macdonald kernel*.

²About the Bessel kernel, see [F, NS, NW, TW2].

³It is well known that the Bessel kernel can be obtained in a scaling limit of the Laguerre kernel, see [F, NS, NW, TW2].

5) The scalar factor *const* in (5.3) can be written in the form

$$const = \frac{2}{\pi^2} (\cos(2\pi\mu) - \cos(2\pi a_0)). \quad (5.4)$$

This expression is periodic in a_0 with period 1. It is strictly positive (because of the assumptions on the parameters). When μ is fixed, its maximal value, attained at the point $a_0 = 1/2$, is equal to $4 \cos^2(\pi\mu)/\pi^2$.

Proof. Step 1: A transformation of the matrix Whittaker kernel. Recall the expression of the Whittaker function through the confluent hypergeometric function:

$$\begin{aligned} x^{-1/2} e^{x/2} W_{\kappa, \mu}(x) &= \frac{\Gamma(-2\mu)x^\mu}{\Gamma(\frac{1}{2} - \kappa - \mu)} {}_1F_1(\frac{1}{2} - \kappa + \mu; 2\mu + 1; x) \\ &+ \frac{\Gamma(2\mu)x^{-\mu}}{\Gamma(\frac{1}{2} - \kappa + \mu)} {}_1F_1(\frac{1}{2} - \kappa - \mu; -2\mu + 1; x). \end{aligned} \quad (5.5)$$

Let us abbreviate

$$\varphi(x) = x^{-1/2} e^{x/2} W_{a+\frac{1}{2}, \mu}(x), \quad (5.6a)$$

$$\varphi_-(x) = x^{-1/2} e^{x/2} W_{a-\frac{1}{2}, \mu}(x), \quad (5.6b)$$

$$\psi(x) = x^{-1/2} e^{x/2} W_{-a+\frac{1}{2}, \mu}(x), \quad (5.6c)$$

$$\psi_-(x) = x^{-1/2} e^{x/2} W_{-a-\frac{1}{2}, \mu}(x), \quad (5.6d)$$

$$\tilde{\varphi}(x) = x\varphi'(x),$$

$$\tilde{\psi}(x) = x\psi'(x).$$

From (5.5), (5.6a,b,c,d) and the series expansion

$${}_1F_1(\alpha; \gamma; x) = \sum_{m \geq 0} \frac{(\alpha)_m}{m!(\gamma)_m} x^m$$

we readily get

$$\begin{aligned} \varphi_-(x) &= \frac{\tilde{\varphi}(x) - a\varphi(x)}{zz'}, \\ \psi_-(x) &= \frac{\tilde{\psi}(x) + a\psi(x)}{zz'}. \end{aligned}$$

Together with the definition of the matrix Whittaker kernel this implies

$$\begin{aligned} e^{\frac{x+y}{2}} K_{++}(x, y) &= \frac{1}{\Gamma(z)\Gamma(z')} \frac{\varphi(x)\varphi_-(y) - \varphi_-(x)\varphi(y)}{x-y} \\ &= \frac{1}{\Gamma(z)\Gamma(z')zz'} \frac{\varphi(x)\tilde{\varphi}(y) - \tilde{\varphi}(x)\varphi(y)}{x-y}, \end{aligned} \quad (5.7a)$$

$$\begin{aligned} e^{\frac{x+y}{2}} K_{--}(x, y) &= \frac{1}{\Gamma(-z)\Gamma(-z')} \frac{\psi(x)\psi_-(y) - \psi_-(x)\psi(y)}{x-y} \\ &= \frac{1}{\Gamma(-z)\Gamma(-z')zz'} \frac{\psi(x)\tilde{\psi}(y) - \tilde{\psi}(x)\psi(y)}{x-y}, \end{aligned} \quad (5.7b)$$

$$\begin{aligned}
e^{\frac{x+y}{2}} K_{+-}(x, y) &= \frac{\sqrt{\sin(\pi z) \sin(\pi z')}}{\pi} \frac{\varphi(x)\psi(y) + zz'\varphi_-(x)\psi_-(y)}{x+y} \\
&= \frac{\sqrt{\sin(\pi z) \sin(\pi z')}}{\pi} \\
&\times \frac{-\frac{a}{zz'}(\varphi(x)\tilde{\psi}(y) - \tilde{\varphi}(x)\psi(y)) + \left(1 - \frac{a^2}{zz'}\right)\varphi(x)\psi(y) + \frac{1}{zz'}\tilde{\varphi}(x)\tilde{\psi}(y)}{x+y}. \quad (5.7c)
\end{aligned}$$

Step 2: The scaling limit of the functions φ , $\tilde{\varphi}$, ψ , $\tilde{\psi}$. We start with the well-known limit formula, which is readily obtained from the standard series expansions for ${}_1F_1$ and ${}_0F_1$:

$$\lim_{|\alpha| \rightarrow \infty} {}_1F_1\left(\alpha; \gamma; \frac{\xi}{\alpha}\right) = {}_0F_1(\gamma; \xi). \quad (5.8)$$

Here α , γ , ξ are allowed to be any complex numbers with the only restriction $\gamma \neq 0, -1, -2, \dots$. The convergence is uniform on compact sets in the ξ -plane, which implies that this limit relation can be differentiated with respect to ξ .

It follows that

$$\begin{aligned}
&\frac{\Gamma(-2\mu)x^\mu}{\Gamma(-a-\mu)} {}_1F_1(-a+\mu; 2\mu+1; x) \\
&\sim \frac{\Gamma(-2\mu)\xi^\mu}{\Gamma(-a-\mu)N^\mu} {}_0F_1(2\mu+1; -\xi) \\
&= \frac{\Gamma(-2\mu)\Gamma(2\mu+1)}{\Gamma(-a-\mu)N^\mu} \frac{\xi^\mu}{\Gamma(2\mu+1)} {}_0F_1(2\mu+1; -\xi) \\
&\sim \Gamma(a+1) \frac{\sin(\pi z)}{\sin(2\pi\mu)} J_{2\mu}(2\sqrt{\xi}). \quad (5.9)
\end{aligned}$$

Here we have used the relations $z = a + \mu$, $a \sim N$,

$$\Gamma(-w)\Gamma(1+w) = -\frac{\pi}{\sin(\pi w)}, \quad \frac{\Gamma(N + \text{const}_1)}{\Gamma(N + \text{const}_2)} \sim N^{\text{const}_1 - \text{const}_2}$$

and the expression of the Bessel function J_ν through the ${}_0F_1$ function.

Likewise,

$$\begin{aligned}
&\frac{\Gamma(-2\mu)x^\mu}{\Gamma(a-\mu)} {}_1F_1(a+\mu; 2\mu+1; x) \\
&\sim \frac{\Gamma(-2\mu)\Gamma(2\mu+1)}{\Gamma(a-\mu)N^\mu} \frac{\xi^\mu}{\Gamma(2\mu+1)} {}_0F_1(2\mu+1; \xi) \\
&\sim -\frac{1}{\Gamma(a)} \frac{\pi}{\sin(2\pi\mu)} I_{2\mu}(2\sqrt{\xi}). \quad (5.10)
\end{aligned}$$

By the definition (5.6a) of the function $\varphi(x)$ and the expression (5.5) for the Whittaker function, $\varphi(x)$ is equal to the left-hand side of (5.9) plus the symmetric expression obtained by inserting $-\mu$ in place of μ . Then it follows from (5.9) that

$$\begin{aligned}
\varphi(x) &\sim \Gamma(a+1) \left(\frac{\sin(\pi z)}{\sin(2\pi\mu)} J_{2\mu}(2\sqrt{\xi}) + \frac{\sin(\pi z')}{\sin(-2\pi\mu)} J_{-2\mu}(2\sqrt{\xi}) \right) \\
&= \Gamma(a+1) A(\xi). \quad (5.11a)
\end{aligned}$$

Here we have used the definition of $A(\xi)$ and the fact that

$$a - \mu = z', \quad \sin(\pi z) = \sin(\pi z_0 + \pi N) = \sin(\pi z_0), \quad \sin(\pi z') = \sin(\pi z'_0)$$

because $N \in 2\mathbb{Z}$ by assumption.

Likewise, it follows from (5.6b), (5.5) and (5.10) that

$$\begin{aligned} \psi(x) &\sim -\frac{1}{\Gamma(a)} \left(\frac{\pi}{\sin(2\pi\mu)} I_{2\mu}(2\sqrt{\xi}) + \frac{\pi}{\sin(-2\pi\mu)} J_{-2\mu}(2\sqrt{\xi}) \right) \\ &= \frac{2}{\Gamma(a)} K_{2\mu}(2\sqrt{\xi}) = \frac{2}{\Gamma(a)} B(\xi). \end{aligned} \quad (5.11b)$$

Recall that our asymptotic formulas, which are based on the limit formula (5.8), are stable under differentiation and note that the differential operator $x \frac{d}{dx}$ is invariant relative to the change of a variable $x \mapsto \xi = xN$. It follows that

$$\tilde{\varphi}(x) = \left(x \frac{d}{dx} \right) \varphi(x) \sim \Gamma(a+1) \left(\xi \frac{d}{d\xi} \right) A(\xi) = \Gamma(a+1) \tilde{A}(\xi), \quad (5.11c)$$

$$\tilde{\psi}(x) = \left(x \frac{d}{dx} \right) \psi(x) \sim \frac{2}{\Gamma(a)} \left(\xi \frac{d}{d\xi} \right) B(\xi) = \frac{2}{\Gamma(a)} \tilde{B}(\xi). \quad (5.11d)$$

Step 3: The scaling limit of the matrix Whittaker kernel. It remains to combine the formulas (5.7a,b,c) with the asymptotic expressions (5.11a,b,c,d).

First of all, note that the transformation of a kernel in x, y under a scaling involves the transformation of a differential, say, dy . We have

$$\frac{dy}{x \pm y} = \frac{d\eta}{\xi \pm \eta},$$

so that in the scaling limit, the denominator $x \pm y$ simply turns into $\xi \pm \eta$. Next, in the scaling limit, the exponential factor in the left-hand side of the formulas (5.7a,b,c) is negligible.

Inserting the asymptotic expressions for φ and $\tilde{\varphi}$ into (5.7a) and using the relation

$$\frac{\Gamma(a+1)\Gamma(a+1)}{\Gamma(z)\Gamma(z')zz'} \sim 1$$

we get the desired formula (5.1a).

Likewise, inserting the asymptotic formulas for ψ and $\tilde{\psi}$ into (5.7b) and using the relation

$$\begin{aligned} \frac{4}{\Gamma(-z)\Gamma(-z')zz'\Gamma(a)\Gamma(a)} &\sim \frac{4 \sin(\pi z) \sin(\pi z')}{\pi^2} \\ &= \frac{4 \sin(\pi z_0) \sin(\pi z'_0)}{\pi^2} \end{aligned}$$

we get the formula (5.1d).

Now, let us examine the numerator in (5.7c),

$$-\frac{a}{zz'} [\varphi(x)\tilde{\psi}(y) - \tilde{\varphi}(x)\psi(y)] + \left(1 - \frac{a^2}{zz'} \right) [\varphi(x)\psi(y)] + \frac{1}{zz'} [\tilde{\varphi}(x)\tilde{\psi}(y)]. \quad (5.12)$$

Note that each of the functions $\varphi(\cdot)$, $\tilde{\varphi}(\cdot)$ is asymptotically equivalent to a function in a scaled variable times the factor $\Gamma(a+1)$, while each of the function $\psi(\cdot)$, $\tilde{\psi}(\cdot)$ is asymptotically equivalent to a function in a scaled variable times the factor $1/\Gamma(a)$. It follows that each of the three expressions in the squared brackets behaves as a function in the scaled variables ξ, η times the factor $\Gamma(a+1)/\Gamma(a) \sim N$.

Further, the coefficients behave as follows

$$\frac{a}{zz'} = \frac{a}{a^2 - \mu^2} \sim N^{-1}, \quad 1 - \frac{a^2}{zz'} = 1 - \frac{a^2}{a^2 - \mu^2} = O(N^{-2}), \quad \frac{1}{zz'} = \frac{1}{a^2 - \mu^2} = O(N^{-2}).$$

This implies that the second and the third summands in (5.12) are asymptotically negligible. It is readily verified that the contribution of the first summand yields the desired formula (5.1b).

Finally, the relation (5.1c) is immediate from the similar relation between the blocks of the matrix Whittaker kernel.

This concludes the proof. \square

REFERENCES

- [DVJ] D. J. Daley, D. Vere-Jones, *An introduction to the theory of point processes*, Springer series in statistics, Springer, 1988.
- [E1] A. Erdelyi (ed.), *Higher transcendental functions*, Vols 1–2, McGraw–Hill, 1953.
- [E2] A. Erdelyi (ed.), *Tables of integral transforms*, Vol. 1, McGraw–Hill, 1954.
- [F] P. J. Forrester, *The spectrum edge of random matrix ensembles*, Nuclear Physics B **402** (1993), no. 3, 709–728.
- [MTW] B. M. McCoy, C. A. Tracy, and T. T. Wu, *Painlevé functions of the third kind*, Jour. Math. Phys. **18** (1977), no. 5, 1058–1092.
- [NS] T. Nagao and K. Slevin, *Nonuniversal correlations for random matrix ensembles*, J. Math. Phys. **34** (1992), no. 5, 2075–2330.
- [NW] T. Nagao and M. Wadati, *Eigenvalue distribution of random matrices at the spectrum edge*, J. Phys. Soc. Japan **62** (1993), no. 11, 3845–3856.
- [P.I] G. Olshanski, *Point processes and the infinite symmetric group. Part I: The general formalism and the density function*, Preprint, 1998, math.RT/9804086.
- [P.II] A. Borodin, *Point processes and the infinite symmetric group. Part II: Higher correlation functions*, Preprint, 1998, math.RT/9804087.
- [P.III] A. Borodin and G. Olshanski, *Point processes and the infinite symmetric group. Part III: Fermion point processes*, Preprint, 1998, math.RT/9804088.
- [P.IV] A. Borodin, *Point processes and the infinite symmetric group. Part IV: Matrix Whittaker kernel*, Preprint, 1998 math/9810013.
- [P.VI] A. Borodin and G. Olshanski, *Point processes and the infinite symmetric group. Part VI: Summary of results*, Preprint, 1998, math/9810015.
- [T] C. A. Tracy, *Asymptotics of a τ -function arising in the two-dimensional Ising model*, Comm. Math. Phys. **142** (1991), no. 2, 297–311.
- [TW1] C. A. Tracy and H. Widom, *Level spacing distributions and the Airy kernel*, Comm. Math. Phys. **159** (1994), 151–174.
- [TW2] ———, *Level spacing distributions and the Bessel kernel*, Comm. Math. Phys. **161** (1994), 289–309.
- [TW3] ———, *Fredholm determinants, differential equations and matrix models*, Comm. Math. Phys. **163** (1994), 33–72.
- [TW4] ———, *Asymptotics of a class of Fredholm determinants*, solv-int/9801008.
- [W] J. Wimp, *A class of integral transforms*, Proc. Edinburgh Math. Soc. **14** (1964), 33–40.